Bipartite Perfect Matching, (Non)Commutative Rank, and Entanglement Transformation

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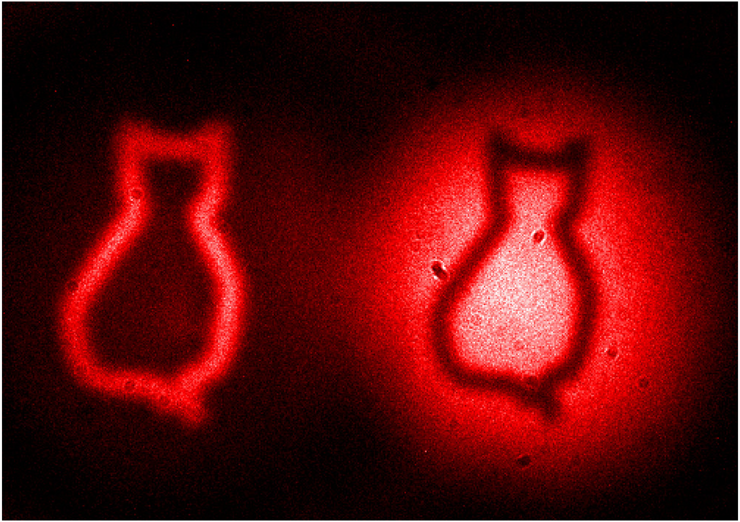
Based on Joint work with Youming Qiao (UTS), Xin Wang and Runyao Duan (Baidu Inc.)

Centrum Wiskunde & Informatica

Research Center for Quantum Software

Quantum Entanglement



From National Geographic.

Quantum Entanglement

Quantum state in a k-partite quantum system: $|\boldsymbol{\Psi}\rangle \in \otimes_{i=1}^{k} \mathbb{C}^{n_i}$, $|||\boldsymbol{\Psi}\rangle||_2 = 1$.

$|\boldsymbol{\Psi}\rangle \in \otimes_{i=1}^{k} \mathbb{C}^{n_i}$ is entangled: $|\boldsymbol{\Psi}\rangle \neq |\psi_1\rangle \otimes \cdots \otimes |\psi_k\rangle$ where $|\psi_i\rangle \in \mathbb{C}^{n_i}$.

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- Qubits: $|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = e_1 \in \mathbb{C}^2$, $|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = e_2 \in \mathbb{C}^2$.
- EPR (Einstein-Podolsky-Rosen) state:

$$|\mathrm{EPR}_2\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle) = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \in \mathbb{C}^2 \otimes \mathbb{C}^2.$$

$$|\mathrm{EPR}_n\rangle = \frac{1}{\sqrt{n}}\sum_{i=0}^{n-1}|i\rangle \otimes |i\rangle \in \mathbb{C}^n \otimes \mathbb{C}^n.$$

- GHZ (Greenberger–Horne–Zeilinger) state:

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- W state:

$$|\mathrm{W}\rangle = \frac{1}{\sqrt{3}}(|1\rangle \otimes |0\rangle \otimes |0\rangle + |0\rangle \otimes |1\rangle \otimes |0\rangle + |0\rangle \otimes |0\rangle \otimes |1\rangle) \in \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$$

Entanglement Transformation

$|\Psi\rangle \in \otimes_{i=1}^{k}\mathbb{C}^{n_i}$ can be **transformed** to $|\Phi\rangle$ if there exist $T_i \in M_{n_i}$, s.t. $|\Phi\rangle = (T_1 \otimes \cdots \otimes T_k)|\Psi\rangle$. ($|\Psi\rangle \xrightarrow{\text{SLOCC}} |\Phi\rangle$).

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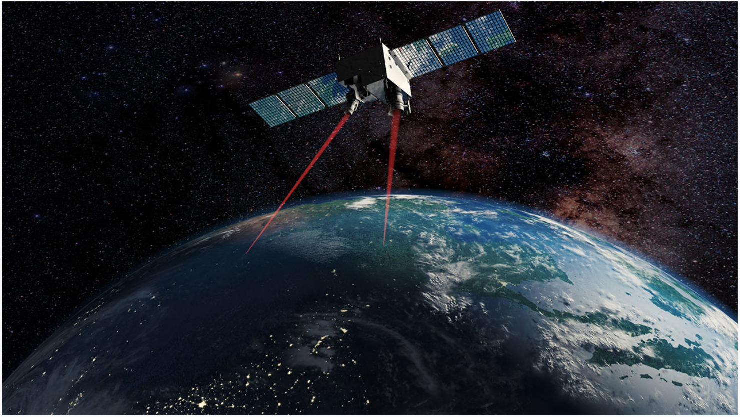

From Science

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Determine bipartite entanglement transformation is "easy":

$$|\Psi\rangle \in \mathbb{C}^n \otimes \mathbb{C}^n \Leftrightarrow A_\Psi \in M_n \quad (|i\rangle \otimes |j\rangle \in \mathbb{C}^n \otimes \mathbb{C}^n \Leftrightarrow E_{i,j} \in M_n)$$

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Determine tripartite entanglement transformation is "hard":

For $|\Psi\rangle \in \mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n$, $|\mathrm{GHZ}_k\rangle \xrightarrow{\text{SLOCC}} |\Psi\rangle \Leftrightarrow \text{tensor rank}(|\Psi\rangle) \leq k$!

Asymptotic Transformation

Examples:

- $|\mathrm{W}\rangle \xrightarrow{\text{SLOCC}} |\mathrm{GHZ}_2\rangle$, $|\mathrm{GHZ}_2\rangle \overset{\text{SLOCC}}{\not\longrightarrow} |\mathrm{W}\rangle$ (Dür, Vidal, Cirac, PRL, 2000)

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The **matrix multiplication exponent**: $\omega = 1/R(|\mathrm{GHZ}_2\rangle, |\Phi^3\rangle)$

$O(n^{\omega})$: time-complexity of multiplying $n \times n$ matrices (over $\mathbb{C}$)

$|\Phi^3\rangle = |\mathrm{EPR}_2\rangle_{AB} \otimes |\mathrm{EPR}_2\rangle_{AC} \otimes |\mathrm{EPR}_2\rangle_{BC} \in \mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4$

Our Result

Given $|\Psi\rangle \in \mathbb{C}_A^n \otimes \mathbb{C}_B^n \otimes \mathbb{C}_C^m$, determine if $|\Psi\rangle_{ABC} \xrightarrow{\text{SLOCC}} |\mathrm{EPR}_n\rangle_{AB} \otimes |0\rangle_C$.

Physical Scenario:

- EPR states are useful and hard to prepare.
- Entanglement of Assistance: Use "noisy" tripartite entanglement shared by A, B and C to generate "noiseless" bipartite entanglement shared by A and B, assisted by C.
- Important to know which "noisy" tripartite entanglement can be used.

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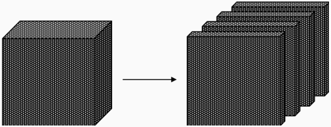
$|\Psi\rangle_{ABC} \in \mathbb{C}^n_A \otimes \mathbb{C}^n_B \otimes \mathbb{C}^m_C \Leftrightarrow$ Matrix space in M_n
$|\Psi\rangle_{ABC} = \sum_{i,j,k} \lambda_{i,j,k} |i\rangle_A \otimes |j\rangle_B \otimes |k\rangle_C \Leftrightarrow \mathcal{A}_\Psi =$
$\mathrm{span}\{A_k = \sum_{i,j} \lambda_{i,j,k} E_{i,j} : k = 1, \ldots, m\}$



Theorem (Chitambar, Duan, Shi, PRA, 2010):

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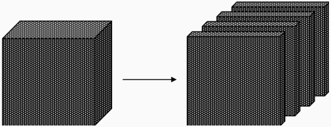

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$\Leftrightarrow$ $\mathcal{A}_\Psi$ has invertible matrices.

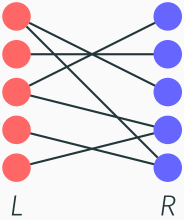
Theorem (Li, Qiao, Wang, Duan, CMP, 2018)

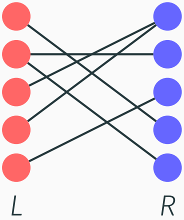
$|\Psi\rangle_{ABC} \xrightarrow{\text{ASLOCC}} |\mathrm{EPR}_n\rangle_{AB} \otimes |0\rangle_C \Leftrightarrow$ *the* **noncommutative** *rank of $\mathcal{A}_\Psi$ is n*

$\Leftrightarrow$ *$\mathcal{A}_\Psi$ has no Shrunk subspace.*

Perfect Matchings in Bipartite Graphs

A **perfect matching** in a bipartite graph $G = (L \cup R, E)$ $(L = R = [n])$ is a subset $M \subseteq E$ s.t. every vertex is incident to exactly one edge in M.





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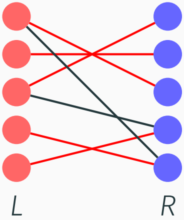

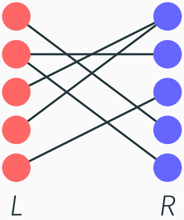

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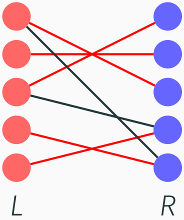

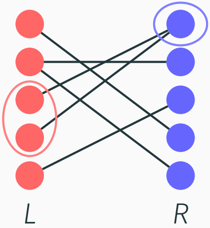

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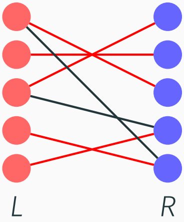
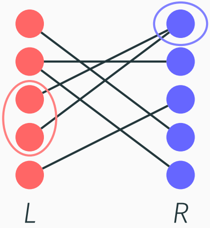


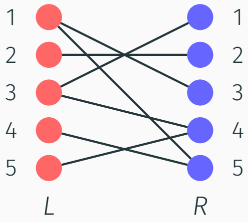


The Marriage Theorem (P. Hall, JLMS, 1935):

G has a P. M. $\Leftrightarrow$ G has no shrunk subset: $S \subseteq L$ such that $|S| > |N(S)|$.

Algebraic Method to Find Perfect Matchings

For $G = ([n] \cup [n], E)$, let $\mathcal{A}_G = \mathrm{span}\{E_{j,i} : (i,j) \in E\} \subseteq M_n$.



$$\mathcal{A}_G = \mathrm{span}\{E_{3,1}, E_{5,1}, E_{2,2}, E_{1,3}, E_{4,3}, E_{5,4}, E_{4,5}\}.$$

$$A = \begin{bmatrix} 0 & 0 & x_{1,3} & 0 & 0 \\ 0 & x_{2,2} & 0 & 0 & 0 \\ x_{3,1} & 0 & 0 & 0 & 0 \\ 0 & 0 & x_{4,3} & 0 & x_{4,5} \\ x_{5,1} & 0 & 0 & x_{5,4} & 0 \end{bmatrix} \in \mathcal{A}_G$$

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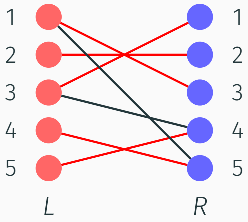

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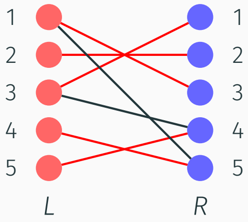

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Randomized poly-time algorithm: Randomly pick a matrix A in $\mathcal{A}_G$, check if A is invertible. (Schwartz-Zippel lemma)

Commutative Rank and Applications in TCS

$\mathcal{A} = \mathrm{span}\{A_1, \ldots, A_m\} \Leftrightarrow$ Symbolic matrix $T = \sum_{i=1}^{m} x_i A_i \in M_n(\mathbb{C}[x_1, \ldots, x_m])$

$\mathbb{C}[x_1, \ldots, x_m]$: The algebra of polynomials in commuting variables $x_1, \ldots, x_m$.

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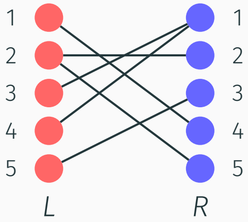
Recall: G has Shrunk subset $\Leftrightarrow$ G has no perfect matching $\Leftrightarrow$ $\mathcal{A}_G$ contains no invertible matrix.

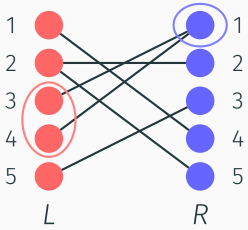
Is there a "singularity witness" for general matrix spaces?



$$\mathcal{A}_G = \mathrm{span}\{E_{4,1}, E_{2,2}, E_{5,2}, E_{1,3}, E_{1,4}, E_{3,5}\}.$$

$$A = \begin{bmatrix} 0 & 0 & x_{1,3} & x_{1,4} & 0 \\ 0 & x_{2,2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_{3,5} \\ x_{4,1} & 0 & 0 & 0 & 0 \\ 0 & x_{5,2} & 0 & 0 & 0 \end{bmatrix} \in \mathcal{A}_G$$

Shrunk Subspaces



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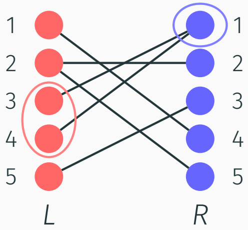

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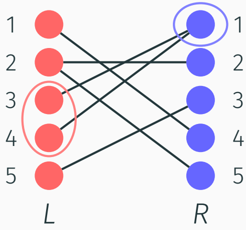

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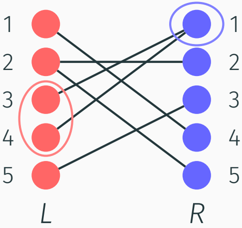

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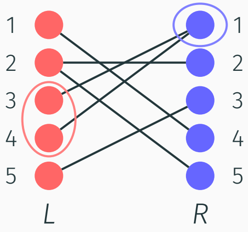

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Not true in general: $\mathcal{A} = \mathrm{span}\left\{ \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \right\}.$

Noncommutative Rank and Recent Progress

$\mathcal{A} = \mathrm{span}\{A_1, \ldots, A_m\} \;\Leftrightarrow\;$ Symbolic matrix $T = \sum_{i=1}^{m} x_i A_i \in M_n(\mathbb{C}\langle x_1, \ldots, x_m \rangle)$

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- **Deterministic** polynomial-time algorithms! (Garg, Gurvits, Oliveira, Wigderson, FOCS, 2016; Ivanyos, Qiao, Subrahmanyam, CC, 2016, 2017).

Proof Outline

Theorem (Li, Qiao, Wang, Duan, CMP, 2018)

$$|\Psi\rangle_{ABC} \xrightarrow{\text{ASLOCC}} |\text{EPR}_n\rangle_{AB} \otimes |0\rangle_C \Leftrightarrow \text{ncrk}\,(\mathcal{A}_\Psi) = n$$

$$\Leftrightarrow \mathcal{A}_\Psi \textit{ has no Shrunk subspace}.$$

$$|\Psi\rangle_{ABC} = \textstyle\sum_{i,j,k} \lambda_{i,j,k} |i\rangle_A \otimes |j\rangle_B \otimes |k\rangle_C \Leftrightarrow \mathcal{A}_\Psi = \text{span}\{A_k = \sum_{i,j} \lambda_{i,j,k} E_{i,j}\}_{k=1,\ldots,m}$$

$$|\Psi\rangle_{ABC} \xrightarrow{\text{ASLOCC}} |\text{EPR}_n\rangle_{AB} \otimes |0\rangle_C \;\Leftrightarrow\; R(|\Psi\rangle_{ABC}, |\text{EPR}_n\rangle_{AB} \otimes |0\rangle_C) = 1$$

$$R(|\Psi\rangle_{ABC}, |\text{EPR}_n\rangle_{AB} \otimes |0\rangle_C) = \sup\{\frac{\ell}{k} : |\Psi\rangle_{ABC}^{\otimes k} \xrightarrow{\text{SLOCC}} (|\text{EPR}_n\rangle_{AB} \otimes |0\rangle_C)^{\otimes \ell}\}$$

$$= \log_n \sup_k \sqrt[k]{\text{crk}\,(\mathcal{A}_\Psi^{\otimes k})}$$

$$= \log_n \lim_{k\to\infty} \sqrt[k]{\text{crk}\,(\mathcal{A}_\Psi^{\otimes k})} := \log_n \text{crk}^{\,\infty}(\mathcal{A}_\Psi)$$

Proof Idea:

- For $\text{ncrk}\,(\mathcal{A}) = n$ ($\mathcal{A}$ has no shrunk subspace), Show $\text{crk}^{\,\infty}(\mathcal{A}) = n$.
- For $\text{ncrk}\,(\mathcal{A}) < n$ ($\mathcal{A}$ has shrunk subspaces), upper bound $\text{crk}^{\,\infty}(\mathcal{A}_\Psi)$.

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$\frac{1}{2}\operatorname{ncrk}(\mathcal{A}) \le \operatorname{crk}(\mathcal{A}) \le \operatorname{ncrk}(\mathcal{A})$ (Fortin, Reutenauer, SLC, 2004).

For $\mathcal{A} \subseteq M_n$ with $\operatorname{ncrk}(\mathcal{A}) = n$ ($\mathcal{A}$ has no shrunk subspace):

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The proof requires the following characterization of shrunk subspaces.

Theorem (Derksen, Weyman, JAMS, 2000; Bürgin,Draisma, MZ, 2006):

$\mathcal{A} \subseteq M_n$ has no shrunk subspace $\Leftrightarrow \exists k \in \mathbb{N}$, s.t. $\mathrm{crk}(\mathcal{A} \otimes M_k) = nk$.

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Let $p = \dim(\mathcal{A}(U))$, $n - q = \dim(U)$, then $\exists\ T_1, T_2 \in \mathrm{GL}_n$, s.t. $\forall\ A \in \mathcal{A}$,

$$T_1 A T_2^t = \left[\begin{array}{c|c} A_{1,1} & A_{1,2} \\ \hline A_{2,1} & 0_{(n-p)\times(n-q)} \end{array}\right]_{n \times n}$$

$\mathcal{A}(p, q, n)$: The minimal space containing all matrices of the above shape.

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- $U \subseteq \mathbb{C}^n$ is a shrunk subspace of $\mathcal{A}$ if $\dim(U) > \dim(\mathcal{A}(U))$

Let $p = \dim(\mathcal{A}(U))$, $n - q = \dim(U)$, then $\exists\, T_1, T_2 \in \mathrm{GL}_n$, s.t. $\forall\, A \in \mathcal{A}$,

$$T_1 A T_2^t = \left[\begin{array}{c|c} A_{1,1} & A_{1,2} \\ \hline A_{2,1} & 0_{(n-p)\times(n-q)} \end{array}\right]_{n \times n}$$

$\mathcal{A}(p, q, n)$: The minimal space containing all matrices of the above shape.

$\operatorname{crk}(\mathcal{A}) = \operatorname{crk}(T_1 \mathcal{A} T_2^t) \leq \operatorname{crk}(\mathcal{A}(p, q, n))$.

Matrix Spaces with Shrunk Subspaces

Proposition:

If $\mathcal{A} \subseteq M_n$ has shrunk subspaces, $\exists 0 \leq \alpha < 1$ s.t. $\mathrm{crk}^{\infty}(\mathcal{A}) \leq \alpha n$.

Recall:

- For $T_1, T_2 \in \mathrm{GL}_n$, $\mathrm{crk}(\mathcal{A}) = \mathrm{crk}(T_1 \mathcal{A} T_2)$.
- $U \subseteq \mathbb{C}^n$ is a shrunk subspace of $\mathcal{A}$ if $\dim(U) > \dim(\mathcal{A}(U))$

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$\mathcal{A}(p, q, n)$: The minimal space containing all matrices of the above shape.

$\mathrm{crk}(\mathcal{A}) = \mathrm{crk}(T_1 \mathcal{A} T_2^t) \leq \mathrm{crk}(\mathcal{A}(p, q, n))$.

Theorem (Li, Qiao, Wang, Duan, CMP, 2018):

$\mathrm{crk}^{\infty}(\mathcal{A}(p, q, n)) = \alpha n$ for some $\alpha < 1$ determined by p, q, n and $p + q < n$.

Concluding Remarks

Matrix spaces arise naturally in Quantum Information and Group Theory.

View these matrix spaces as generalizations of graphs.