The Haemers bound of Graphs and Noncommutative Graphs, and quantum Shannon Capacities

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Based on joint works with Jeroen Zuiddam (IAS) and Sander Gribling (CWI & QuSoft)

Centrum Wiskunde & Informatica

Research Center for Quantum Software

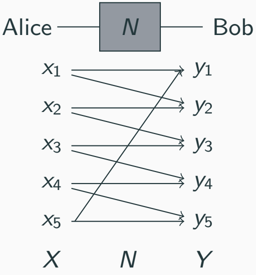
Main Results

- The Haemers bound over $\mathbb{R}$ upper bound the zero-error capacity of classical channels assisted by maximally entangled state and projective measurements.
- There is a noncommutative Haemers bound which upper bounds the zero-error classical capacity of quantum channels.

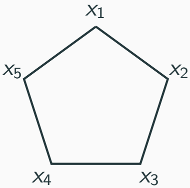
The Zero-error Capacity of Classical Channels

Zero-error Communication through Classical Channels

Classical communication



Confusability graph



$x_i \sim x_j$ if $\exists y \in Y$, s.t.
$N(y|x_i)N(y|x_j) > 0$.

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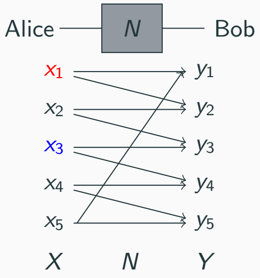

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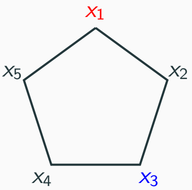

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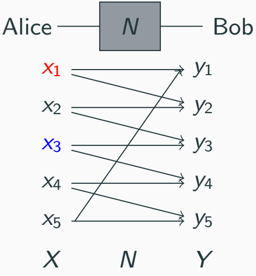

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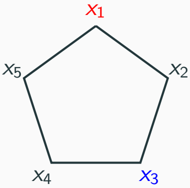

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- Zero-error encoding of $N \Leftrightarrow$ Independent set of G_N
- Maximum # zero-error messages send through N: $\alpha(G_N)$

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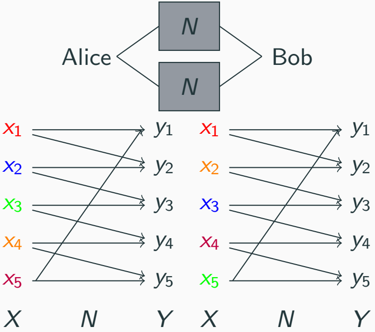

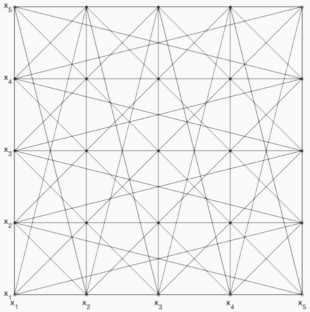

$\{\{(g,h),(g',h')\} \in E(G \boxtimes H)$ if

- $g = g',\ \{h,h'\} \in E(H)$ or
- $\{g,g'\} \in E(G),\ h = h'$ or
- $\{g,g'\} \in E(G),\ \{h,h'\} \in E(H)$

- Zero-error encoding of N $\Leftrightarrow$ Independent set of G_N
- Maximum # zero-error messages send through N: $\alpha(G_N)$
- Block-code of length k through N $\Leftrightarrow$ confusability graph $G_N^{\boxtimes k}$

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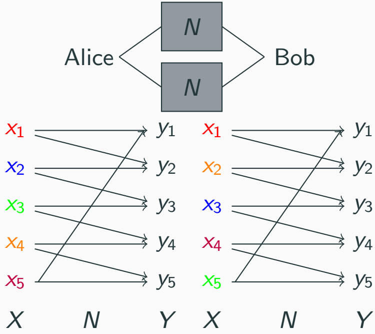

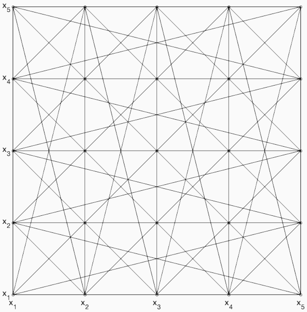

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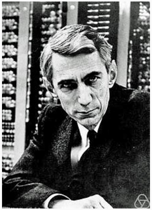
- Maximum # zero-error messages send through N: $\alpha(G_N)$
- Block-code of length k through N $\Leftrightarrow$ confusability graph $G_N^{\boxtimes k}$
- Shannon capacity [Shannon 1956] of G_N (N):

$$\Theta(G_N) := \sup_k \sqrt[k]{\alpha(G_N^{\boxtimes k})} = \lim_{k\to\infty} \sqrt[k]{\alpha(G_N^{\boxtimes k})}$$



Asymptotic Spectrum of Graphs

The sum of channels corresponds physically to a situation where either of two channels may be used (but not both), a new choice being made for each transmitted letter. The product channel corresponds to a situation where both channels are used each unit of time. It is interesting to note that multiplication and addition of channels are both associative and commutative, and that the product distributes over a sum. Thus one can develop a kind of algebra for channels in which it is possible to write, for example, a polynomial $\sum a_n K^n$, where the a_n are non-negative integers a and K is a channel. We shall not, however, investigate here the algebraic properties of this system.

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Classical channels $\Leftrightarrow$ Isomorphism classes of graphs.
Sum of two channels $N + M$ $\Leftrightarrow$ Disjoint union $G_N \sqcup G_M$
Product of two channels $N \times M$ $\Leftrightarrow$ Strong product $G_N \boxtimes G_M$

Comm. semiring $\mathcal{G} = (\{\text{Iso. cl. of graphs}\}, \sqcup, \boxtimes, K_0, K_1)$

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$$G \leq H \;\Leftrightarrow\; \exists f : V(G) \to V(H), \text{ s.t. } \{g, g'\} \notin E(G) \;\Rightarrow\; \{f(g), f(g')\} \notin E(H)$$

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Asymptotic spectrum of graphs $\mathbf{X}(\mathcal{G}) := \{\leq$-monotone **homomorphisms** $\phi : \mathcal{G} \to \mathbb{R}_{\geq 0}\}$

- Normalized: $\phi(K_1) = 1$
- Additive: $\phi(G \sqcup H) = \phi(G) + \phi(H)$
- Multiplicative: $\phi(G \boxtimes H) = \phi(G)\phi(H)$
- Monotone: $G \leq H \Rightarrow \phi(G) \leq \phi(H)$

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For any $\phi \in \mathbf{X}(\mathcal{G})$, $\phi(\overline{K}_d) = \phi(\underbrace{K_1 \sqcup \cdots \sqcup K_1}_{d}) = d$, $\overline{K}_d \leq G \leq \overline{K}_{d'} \Rightarrow d \leq \phi(G) \leq d'$.

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For any $\phi \in \mathbf{X}(\mathcal{G})$, $\Theta(G) \leq \phi(G) \leq \inf_k \sqrt[k]{\overline{\chi}(G^{\boxtimes k})} = \overline{\chi}_f(G)$.

$\Theta(G) \leq \min\{\phi(G) : \phi \in \mathbf{X}(\mathcal{G})\} \leq \max\{\phi(G) : \phi \in \mathbf{X}(\mathcal{G})\} \leq \overline{\chi}_f(G)$

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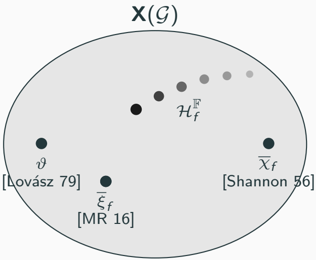
$\Theta(G) = \min\{\phi(G) : \phi \in \mathbf{X}(\mathcal{G})\}$, $\overline{\chi}_f(G) = \max\{\phi(G) : \phi \in \mathbf{X}(\mathcal{G})\}$ [Zuiddam 2019]

Remark: $\Theta \notin \mathbf{X}(\mathcal{G})$ and $\overline{\chi}_f \in \mathbf{X}(\mathcal{G})$. The proof follows Strassen's theory of asymptotic spectra.

Spectral points in $\mathbf{X}(\mathcal{G})$



Haemers Bound [Haemers 1979]: For $G = ([n], E)$

$$\mathcal{H}^{\mathbb{F}}(G) = \min\{\mathrm{rank}\,(B) : \; B \in M_n(\mathbb{F}), \; B_{i,i} = 1 \text{ for } i \in [n], \; B_{i,j} = 0 \text{ for } \{i,j\} \notin E\}$$

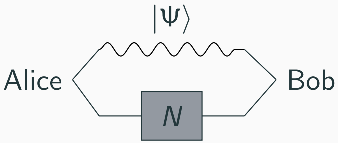
Fractional Haemers bound [Blasiak 2013, Bukh-Cox 2018]:

$$\mathcal{H}_f^{\mathbb{F}}(G) := \inf\{\frac{\mathrm{rank}\,(B)}{d} : \; B \in M_n(M_d(\mathbb{F})), \; B_{i,i} = I_d \; \forall\, i \in [n], \; B_{i,j} = 0 \text{ for } \{i,j\} \notin E\}.$$

- $\mathcal{H}_f^{\mathbb{F}}(G) \leq \mathcal{H}^{\mathbb{F}}(G)$
- $\vartheta, \mathcal{H}_f^{\mathbb{F}}$ are incomparable $\forall\, \mathbb{F}$. $\vartheta, \mathcal{H}_f^{\mathbb{R}} \leq \overline{\xi}_f \leq \overline{\chi}_f = \max\{\phi : \phi \in \mathbf{X}(\mathcal{G})\}$
- $\mathcal{H}_f^{\mathbb{F}}, \mathcal{H}_f^{\mathbb{F}'}$ are different $\forall\, \mathbb{F}, \mathbb{F}'$ of different characteristic [BC18]

Entanglement-assisted Zero-error Capacity and the Haemers Bound

Entanglement-assisted zero-error communication



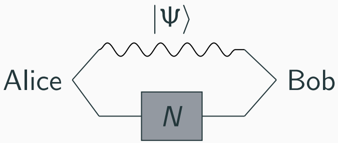
Still characterized by its confusability graph G_N [Cubitt-Leung-Matthews-Winter 2010]

Maximum # zero-error messages send through N assisted by entanglement: $\alpha_*(G_N)$

Entanglement-assisted Shannon capacity:

$$\Theta_*(G_N) = \sup_k \sqrt[k]{\alpha_*(G_N^{\boxtimes k})} = \lim_{k\to\infty} \sqrt[k]{\alpha_*(G_N^{\boxtimes k})}$$

Entanglement-assisted zero-error communication



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A restricted setting [Mančinska-Roberson 2016] (originally from non-local games):

Assisted by maximally entangled state & projective measurements: $\alpha_q(G_N)$

Quantum Shannon capacity:

$$\Theta_q(G_N) = \sup_k \sqrt[k]{\alpha_q(G_N^{\boxtimes k})} = \lim_{k\to\infty} \sqrt[k]{\alpha_q(G_N^{\boxtimes k})}$$

Asymptotic Spectrum Characterizations for Θ_*

Entanglement-assisted homomorphism $G \xrightarrow{*} H$ [Cubitt et al. 2014]:

If $\exists\ \rho > 0$ and $\rho_g^h \geq 0\ \forall\ g \in V(G)\ h \in V(H)$,

- $\forall\ g \in V(G)$, $\sum_{h \in V(H)} \rho_g^h = \rho$
- $\rho_g^h \rho_{g'}^{h'} = 0$ if $\{g, g'\} \in E(G)$ and $\{h, h'\} \notin E(H)$

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$(\phi \in \mathbf{X}(\mathcal{G})$ and $\phi(G) \leq \phi(H)$ if $G \leq_* H)$

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Theorem [Li-Zuiddam 2019]:

$$\Theta_*(G) = \sup_k \sqrt[k]{\alpha_*(G^{\boxtimes k})} = \min\{\phi(G) : \phi \in \mathbf{X}_*(\mathcal{G})\}$$
$$\inf_k \sqrt[k]{\overline{\chi}_*(G^{\boxtimes k})} = \max\{\phi(G) : \phi \in \mathbf{X}_*(\mathcal{G})\}$$

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Quantum homomorphism $G \xrightarrow{q} H$ [Mančinska-Roberson 2016]:

If $\exists$ **projectors** P_g^h $\forall$ $g \in V(G)$ $h \in V(H)$,

- $\forall$ $g \in V(G)$, $\sum_{h \in V(H)} P_g^h = I$
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Asymptotic Spectrum Characterizations for Θ_q

Quantum homomorphism $G \xrightarrow{q} H$ [Mančinska-Roberson 2016]:

If $\exists$ **projectors** $P_g^h \; \forall \; g \in V(G) \; h \in V(H)$,

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Quantum Asymptotic Spectrum $\mathbf{X}_q(\mathcal{G}) := \{\leq_q\text{-monotone}$ homo. $\;\phi : \mathcal{G} \to \mathbb{R}_{\geq 0}\}$

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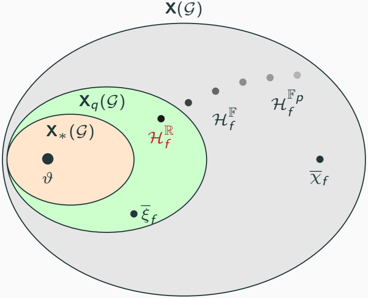
Theorem [Li-Zuiddam 2019]:

$$\Theta_q(G) = \sup_k \sqrt[k]{\alpha_q(G^{\boxtimes k})} = \min\{\phi(G) : \phi \in \mathbf{X}_q(\mathcal{G})\}$$
$$\inf_k \sqrt[k]{\overline{\chi}_q(G^{\boxtimes k})} = \max\{\phi(G) : \phi \in \mathbf{X}_q(\mathcal{G})\}$$

Spectral Points



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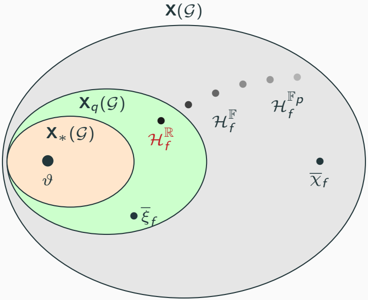

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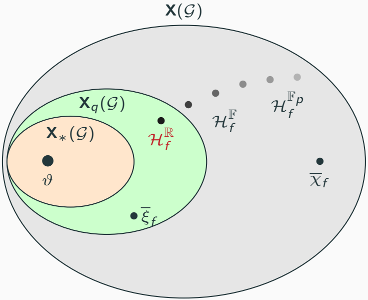

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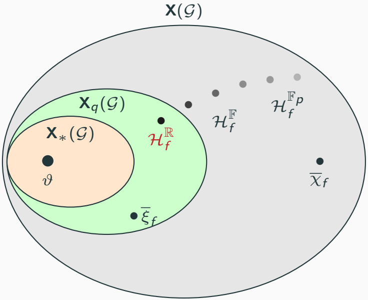

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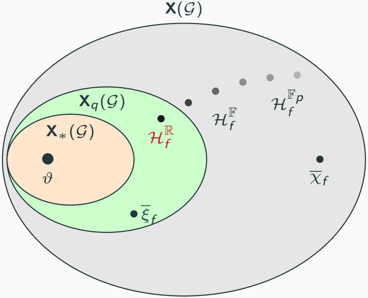

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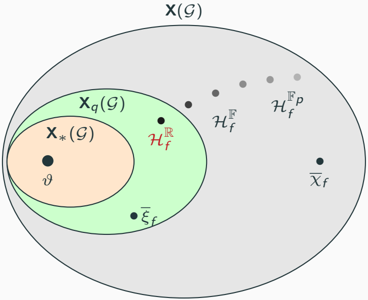

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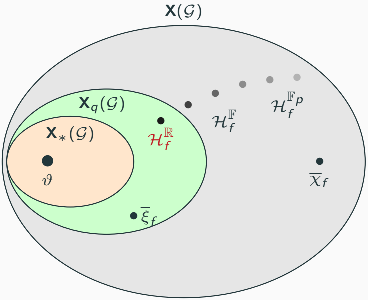

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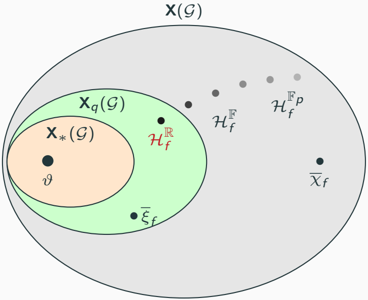

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  Cannot be true at the same time! Since $\Theta_q(G) \leq \Theta_*(G) \leq \vartheta(G)$.

The Zero-error Classical Capacity of Quantum Channels and a Noncommutative Haemers Bound

Zero-error Communication through Quantum Channels

Quantum Channel: $\Phi(\rho) = \sum_{k=1}^{m} E_k \rho E_k^{\dagger}$ $\forall \rho \in M_n$ satisfying $\sum_{k=1}^{m} E_k^{\dagger} E_k = I_n$

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Send classical zero-error messages via Φ: Encode $i \mapsto |\psi_i\rangle\langle\psi_i|$

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$$\Phi(|\psi_i\rangle\langle\psi_i|) \perp \Phi(|\psi_j\rangle\langle\psi_j|) \Leftrightarrow \mathrm{Tr}(\Phi(|\psi_i\rangle\langle\psi_i|)^\dagger \Phi(|\psi_j\rangle\langle\psi_j|)) = 0 \Leftrightarrow \sum_{k,k'=1}^{m} |\langle\psi_i| E_k^\dagger E_{k'} |\psi_j\rangle|^2 = 0$$

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Noncommutative (confusability) graph [Duan-Severini-Winter 2013]:

$$S_\Phi = \mathrm{span}\{E_k^\dagger E_{k'} : \; k, k' = 1, \ldots, m\} \subseteq M_n$$

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- Different Choi-Kraus representations of Φ result in the same S_Φ;
- S_Φ is an operator system ($S_\Phi = S_\Phi^\dagger, I_n \in S_\Phi$). From any operator system S there exists Φ such that $S = S_\Phi$ [Duan 2009, Cubitt-Chen-Harrow 2009]
- Two noncommutative graph S_Φ and S_Ψ are "isomorphic" if there exists a unitary matrix $U \in M_n$ such that $U^\dagger S_\Phi U = S_\Psi$.
- $\Phi \otimes \Psi \ \Rightarrow S_\Phi \otimes S_\Psi$, $\Phi + \Psi \ \Rightarrow S_\Phi \oplus S_\Psi$.

Zero-error Communication through Quantum Channels

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Maximum # zero-error messages send through Φ $\Leftrightarrow$ Independence number of S_Φ

$$\alpha(S_\Phi) = \max\{\ell : \ \exists |\psi_1\rangle, \ldots, |\psi_\ell\rangle, \ \langle\psi_i|A|\psi_j\rangle = 0 \ \forall \ i \neq j \ \& \ A \in S_\Phi\}.$$

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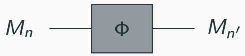

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The Shannon capacity of S_Φ: $\Theta(S_\Phi) := \sup_k \sqrt[k]{\alpha(S_\Phi^{\otimes k})} = \lim_{k\to\infty} \sqrt[k]{\alpha(S_\Phi^{\otimes k})}$

Graphs and Noncommutative Graphs

Classical Channel N	Quantum Channel N
$\{N(j\|i) : i \in [n], j \in [n']\}$	$N(\rho) = \sum_{i=1}^{n} \sum_{j=1}^{n'} N(j\|i) \|j\rangle\langle i\|\rho\|i\rangle\langle j\|$
Confusability graph	Noncommutative graph
$G = ([n], E)$ where $\{i, i'\} \in E$ if $\exists\, j \in [n'],\ N(j\|i)N(j\|i') > 0.$	$S_N = \mathrm{span}\{\|i\rangle\langle i'\| :\ \exists\, j,\ \sqrt{N(j\|i)N(j\|i')} > 0\}$

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$G \mapsto S_G$: Semiring homomorphism $\mathcal{G} \to \mathcal{S} = (\{$Iso. cl. of nc-graphs$\}, \oplus, \otimes, \{0\}, \mathbb{C})$

- $G \cong H \Leftrightarrow S_G \cong S_H$ ($S \cong T \Leftrightarrow \exists U \in U_n,\ U^\dagger S U = T$) [Ortiz-Paulsen 2015]
- $G \boxtimes H \mapsto S_{G \boxtimes H} = S_G \otimes S_H$, $G \sqcup H \mapsto S_{G \sqcup H} = S_G \oplus S_H$.

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- $d \mapsto \overline{K}_d \mapsto \mathcal{D}_d = \mathrm{span}\{|i\rangle\langle i| : i \in [d]\} \subseteq M_d$
- $\alpha(G) = \alpha(S_G)$, $\Theta(G) = \Theta(S_G)$ [Duan-Severini-Winter 2013].

Graphs and Noncommutative Graphs

Classical Channel N	Quantum Channel N
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Confusability graph G	Nonccommutative graph S_G
$G = ([n], E)$ where $\{i, i'\} \in E$ if $\exists\ j \in [n'],\ N(j\|i)N(j\|i') > 0$.	$S_G = \mathrm{span}\{\|i\rangle\langle i'\| :\ i = i' \in [n]$ or $\{i, i'\} \in E\}$

$G \mapsto S_G$: Semiring homomorphism $\mathcal{G} \to \mathcal{S} = (\{\text{Iso. cl. of nc-graphs}\}, \oplus, \otimes, \{0\}, \mathbb{C})$

Preorder on $\mathcal{S}$ [Stahlke 2016]:

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$\overline{\chi}(S) = \min\{d : S \leq \mathcal{D}_d\}$ is not well-defined: $S = \mathrm{span}\{I_2\}$, $\overline{\chi}(S) = +\infty$.

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Open problem: $\Theta(S) =?\ \min\{\phi(S) : \phi \in \mathbf{X}(\mathcal{S})\}$

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Theorem [Li-Zuiddam 2019]: Asymptotic spectrum characterization for E.-A. Shannon capacity of noncommutative graphs.

Upper bounds on the Noncommutative Shannon Capacity

Graph Shannon capacity upper bound

$$\phi : \mathcal{G} \to \mathbb{R}_{\geq 0}$$

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The Lovász theta function [Lovász 1979]:

$$\begin{aligned}\vartheta(G) &= \max\{\|I + T\| : I + T \geq 0,\ T_{i,j} = 0 \text{ if } \{i,j\} \in E \text{ or } i = j\} \\ &= \max\{\|I + T\| : I + T \geq 0, T \in S_G^{\perp}\}\end{aligned}$$

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- $\vartheta(S) \leq \vartheta(T)$ if $S \leq T$.
- $\vartheta : \mathcal{S} \to \mathbb{R}_{\geq 0}$ is only **supermultiplicative**. ($\vartheta : \mathcal{G} \to \mathbb{R}_{\geq 0}$ is multiplicative.)

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$$\tilde{\vartheta}(S) = \sup_m \vartheta(S \otimes M_m) = \sup_m \max\{||I + T|| : I + T \geq 0, T \in S^{\perp} \otimes M_m\}$$

- $\tilde{\vartheta}(S_G) = \vartheta(G)$ for any graph G.
- $\tilde{\vartheta}(S) \leq \tilde{\vartheta}(T)$ if $S \leq T$.
- $\tilde{\vartheta}(S \otimes T) = \tilde{\vartheta}(S)\tilde{\vartheta}(T)$.
- $\tilde{\vartheta}(S) \geq \Theta(S)$ for any noncommutative graph S.
- $\tilde{\vartheta}(S)$ can be computed by semidefinite programming.

The Haemers Bound of Noncommutative Graphs

$$\begin{aligned}\mathcal{H}^{\mathbb{C}}(G) &= \min\{\mathrm{rank}\,(B): \; B \in M_n(\mathbb{C}),\; B_{i,i} = 1 \text{ for } i \in [n],\; B_{i,j} = 0 \text{ for } \{i,j\} \notin E\} \\ &= \min\{\mathrm{rank}\,(I + B): \quad B_{i,j} = 0 \text{ if } \{i,j\} \notin E \text{ or } i = j\} \\ &= \min\{\mathrm{rank}\,(I + B): \quad B \in S_G^{\perp}\}\end{aligned}$$

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[Gribling-Li 2020]: $\mathcal{H}(S) = \min\{\operatorname{rank}(B): \; m \in \mathbb{N},\; B \in M_m(S),\; \sum_{i=1}^{m} B_{i,i} = I_n\}$.

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Proof idea:

- Haemers bound $\mathcal{H}^{\mathbb{C}}(G)$ + positive semidefiniteness = orthogonal rank $\overline{\xi}(G) = \min\{\mathrm{rank}\,(I + B) : \quad B_{i,j} = 0 \text{ if } \{i,j\} \notin E \text{ or } i = j, I + B \geq 0\}$. [Peeters 1996]

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- Remove PSD condition!

The Haemers Bound of Noncommutative Graphs

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Properties of $\mathcal{H}(S)$:

- $\mathcal{H}(S_G) = \mathcal{H}(G)$ for any graph G.
- $\mathcal{H}(S \otimes T) \leq \mathcal{H}(S)\mathcal{H}(T)$, $\mathcal{H}(S \oplus T) = \mathcal{H}(S) + \mathcal{H}(T)$.
- $\mathcal{H}(S) \leq \mathcal{H}(T)$ if $S \leq T$.
- $\Theta(S) \leq \mathcal{H}(S)$ for any noncommutative graph S.
- $\mathcal{H}(S) < \tilde{\vartheta}(S)$ for some noncommutative graph S.

Spectral Points in $\mathbf{X}(\mathcal{S})$

Proposition: There is a surjective map $\eta : \mathbf{X}(\mathcal{S}) \to \mathbf{X}(\mathcal{G})$

Natural Question: Starting from $\phi \in \mathbf{X}(\mathcal{G})$, construct $\phi \in \mathbf{X}(\mathcal{S})$.

Example: $\vartheta \mapsto \tilde{\vartheta}$.

Other known points in $\mathbf{X}(\mathcal{G})$?

- The fractional Haemers bound $\mathcal{H}_f^{\mathbb{C}}(G)$
- The projective rank (fractional orthogonal rank) $\overline{\xi}_f(G)$
- The fractional clique cover number $\overline{\chi}_f(G)$

Open problem: How to get fractional noncommutative graph parameters?

Work in progress: Noncommutative fractional Haemers bound and projective rank. (Don't know how to prove multiplicativity.)


Thank You!