On a tracial version of Haemers bound

Official title: The (fractional) Haemers bound for noncommutative graphs

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SIAM Conference on Optimization, 2021

Based on the joint work (arXiv: 2107.02567) with Li Gao (TUM) and Sander Gribling (IRIF)

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Shannon capacity and asymptotic spectrum of graphs

For a graph $G = ([n], E)$,

- Independence number $\alpha(G)$: size of the largest independent set (empty subgraph).
- Shannon capacity $\Theta(G) := \sup_k \sqrt[k]{\alpha(G^{\boxtimes k})} = \lim_{k \to \infty} \sqrt[k]{\alpha(G^{\boxtimes k})}$
- Characterize the zero-error capacity of classical channels [Shannon 1956]

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Asymptotic spectrum of graphs X: Consists of $\phi : \{\text{Graphs}\} \to \mathbb{R}_{\geq 0}$ satisfying

1. $\phi(\overline{K}_1) = 1$ [Normalization]
2. $\phi(G \boxtimes H) = \phi(G)\phi(H)$ [Multiplicativity]
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4. $\phi(G) \leq \phi(H)$ if $\overline{G} \to \overline{H}$ [Monotonicity]

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4. $\phi(G) \leq \phi(H)$ if $\overline{G} \to \overline{H}$ [Monotonicity]
5. $(1+3) \Rightarrow \phi(\overline{K}_d) = d$ for any d
6. $(4+5) \Rightarrow \alpha(G) = \max\{d : K_d \to \overline{G}\} \leq \phi(G)$

- $(2+6) \Rightarrow \Theta(G) \leq \phi(G)$
- $\Theta(G) \leq \phi(G),\ \forall\, \phi \in \mathsf{X}$

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- The Lovász theta function $\vartheta(G)$
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Theorem [Zuiddam 19]: $\Theta(G) = \min\{\phi(G) : \phi \in \mathbf{X}\}$.

Outline

Goal of this talk:

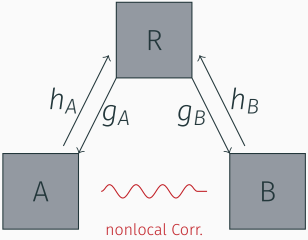
- Introduce quantum versions of independence numbers and Shannon capacities via nonlocal games
- Characterize quantum Shannon capacities in terms of quantum asymptotic spectra of graphs
- New elements in the asymptotic spectrum of graphs: Tracial rank [PSSTW 16] and tracial Haemers bound [GGL 21]

Connections with the symposium topic:

- Characterizations of certain entanglement-assisted zero-error capacities of classical channels
- Many spectrum points can be placed in the framework of tracial polynomial optimization

A Nonlocal Game for Graph Homomorphism

- Alice and Bob know two graphs G, H and share some correlation
- The referee send $g_A, g_B \in V(G)$
- Alice and Bob return $h_A, h_B \in V(H)$
- Winning condition:
  - $g_A = g_B \Rightarrow h_A = h_B$
  - $g_A \sim g_B \Rightarrow h_A \sim h_B$



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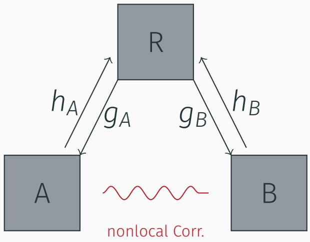

Q.: When do perfect strategy exists?

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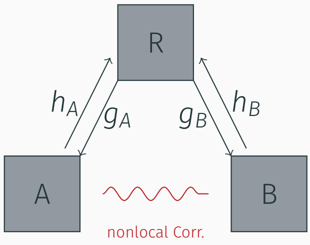

If Alice and Bob share

- **Local randomness**: $\exists$ perfect strategy $\Leftrightarrow$ graph hom. $G \to H$

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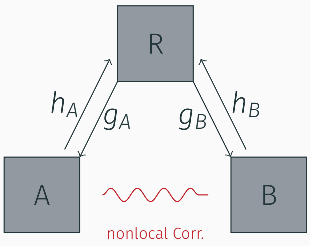

If Alice and Bob share

- **Local randomness**: $\exists$ perfect strategy $\Leftrightarrow$ graph hom. $G \to H$
- **Entanglement in tensor-product model**: $\exists$ perfect strategy $\Leftrightarrow G \xrightarrow{q} H$
- **Entanglement in commuting-operator model**: $\exists$ perfect strategy $\Leftrightarrow G \xrightarrow{qc} H$

$$G \to H \;\Rightarrow\; G \xrightarrow{q} H \;\Rightarrow\; G \xrightarrow{qc} H$$

Quantum Independence Numbers and Shannon Capacities

For a graph $G = ([n], E)$ and $t \in \{q, qc\}$,

- *t*-independence number $\alpha_t(G) := \max\{d : K_d \xrightarrow{t} \overline{G}\}$
- $\alpha_q(G)$: Characterize the largest zero-error encoding of classical channel assisted by maximally entangled state and projective measurement.
  $\alpha_*(G)$: Characterize the largest zero-error encoding of classical channel assisted by entanglement. $\alpha_q \leq \alpha_*$.

Quantum Independence Numbers and Shannon Capacities

For a graph $G = ([n], E)$ and $t \in \{q, qc\}$,

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- t-Shannon capacity $\Theta_t(G) := \sup_k \sqrt[k]{\alpha_t(G^{\boxtimes k})} = \lim_{k\to\infty} \sqrt[k]{\alpha_t(G^{\boxtimes k})}$
- t-asymptotic spectrum of graphs:

$$\mathsf{X}_t = \{\phi \in \mathsf{X} : \phi(G) \leq \phi(H) \text{ if } \overline{G} \xrightarrow{t} \overline{H}\}$$

Thus $\mathsf{X}_{qc} \subseteq \mathsf{X}_q \subseteq \mathsf{X}$.

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Elements in X_q:

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Elements in X_{qc}:

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- The tracial rank $\overline{\xi}_{tr}(G)$
- The tracial Haemers bound $\mathcal{H}_{tr}(G)$

Tracial Rank

λ-tracial representation of $G = ([n], E)$:

- A C^*-algebra $\mathcal{A}$ with a tracial state $\tau : \mathcal{A} \to \mathbb{C}$
- Projections $\{P_i\}_{i \in [n]} \subseteq \mathcal{A}$ s.t. $\tau(P_i) = \frac{1}{\lambda}$ $\forall\, i \in [n]$ and $P_i P_j = 0$ if $\{i, j\} \notin E$

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- Comm. and Noncomm. ψ hierarchies [Gvozdenović-Laurent 08, GdLL 18]
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  - $\vartheta(G) = \xi_1^{\text{col}}(\overline{G}) \leq \cdots \leq \xi_\infty^{\text{col}}(\overline{G}) = \overline{\xi}_{tr}(G) \leq \xi_*^{\text{col}}(G) = \overline{\xi}_f(G)$

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- [GGL 21]: $\overline{\xi}_{tr}(G \sqcup H) = \overline{\xi}_{tr}(G) + \overline{\xi}_{tr}(H)$. $\overline{\xi}_{tr}(\overline{K}_1) = 1$ (trivial)
- Conclusion: $\overline{\xi}_{tr} \in \mathbf{X}_{qc} \subseteq \mathbf{X}_q \subseteq \mathbf{X}$

Tracial Haemers Bound

$\frac{d}{r}$-subspace representation of $G = ([n], E)$ [Introduced by Lex Schrijver]:

dim-r subspaces $S_i \subseteq \mathbb{C}^d \ \forall\, i \in [n]$ such that $S_i \cap (\sum_{j:\{i,j\} \notin E} S_j) = \{0\} \ \forall\, i \in [n]$

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- A von Neumann algebra $\mathcal{M}$ with a tracial state $\tau : \mathcal{M} \to \mathbb{C}$
- Projections $\{P_i\}_{i\in[n]} \subseteq \mathcal{M}$ s.t. $\tau(P_i) = \frac{1}{\lambda}$ and $P_i \wedge (\bigvee_{j:\{i,j\}\notin E} P_j) = 0 \; \forall\, i \in [n]$

Tracial Haemers bound: $\mathcal{H}_{tr}(G) = \inf\{\lambda : \; \exists\, \lambda - \text{tracial subspace rep. of } G\}$.
Notation: For projections $P, Q \in \mathcal{M} \subseteq B(\mathcal{H})$,

- $P \wedge Q \in \mathcal{M}$: Projection onto the **closed subspace** $\mathrm{ran}(P) \cap \mathrm{ran}(Q)$
- $P \vee Q \in \mathcal{M}$: Projection onto the **closed subspace** $\mathrm{cl}(\mathrm{ran}(P) + \mathrm{ran}(Q))$

Remark: $P_i \wedge (\bigvee_{j:\{i,j\}\notin E} P_j) = 0 \;\Leftrightarrow\; \mathrm{ran}(P_i) \cap \mathrm{cl}(\sum_{j:\{i,j\}\notin E} \mathrm{ran}(P_j)) = \{0\}$

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Remark:

- $P_i \wedge (\bigvee_{j:\{i,j\} \notin E} P_j) = 0 \;\Leftrightarrow\; \mathrm{ran}(P_i) \cap \mathrm{cl}(\sum_{j:\{i,j\} \notin E} \mathrm{ran}(P_j)) = \{0\}$
- Von Neumann algebra is necessary. ($\wedge$ and $\vee$ are closed in $\mathcal{M}$)
- λ-tracial rep. ($P_i P_j = 0$ if $\{i,j\} \notin E$) $\Rightarrow$ a λ-tracial subspace rep.

Properties of the Tracial Haemers Bound

$\mathcal{H}_{tr}(G) = \inf\{\lambda : \exists\ \lambda -$ tracial subspace rep. of $G\}$, where a λ-tracial subspace rep. is

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1. $\mathcal{M}$ is Finite-dim. $\Rightarrow \mathcal{H}_f(G;\mathbb{C})$; $\mathcal{M}$ is Comm. $\Rightarrow \overline{\chi}_f(G)$; $\mathcal{H}_{tr} \leq \overline{\xi}_{tr}$
2. $\exists G$ s.t. $\mathcal{H}_{tr}(G) \leq \mathcal{H}_f(G) < \vartheta(G)$ and $\mathcal{H}_{tr}(C_5) = \frac{5}{2} > \vartheta(C_5)$ ($\mathcal{H}_{tr}(C_{2k+1}) = k + \frac{1}{2}$)

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2. $\exists G$ s.t. $\mathcal{H}_{tr}(G) \leq \mathcal{H}_f(G) < \vartheta(G)$ and $\mathcal{H}_{tr}(C_5) = \frac{5}{2} > \vartheta(C_5)$ ($\mathcal{H}_{tr}(C_{2k+1}) = k + \frac{1}{2}$)
3. $\mathcal{H}_{tr}(\overline{K}_d) = d$; $\mathcal{H}_{tr}(G) \leq \mathcal{H}_{tr}(H)$ if $\overline{G} \xrightarrow{qc} \overline{H}$
4. $\mathcal{H}_{tr}(G \boxtimes H) \leq \mathcal{H}_{tr}(G)\mathcal{H}_{tr}(H)$; $\mathcal{H}_{tr}(G \sqcup H) \leq \mathcal{H}_{tr}(G) + \mathcal{H}_{tr}(H)$

Properties of the Tracial Haemers Bound

$\mathcal{H}_{tr}(G) = \inf\{\lambda : \exists\ \lambda - \text{tracial subspace rep. of } G\}$, where a λ-tracial subspace rep. is

- A von Neumann algebra $\mathcal{M}$ with a tracial state $\tau : \mathcal{M} \to \mathbb{C}$
- Projections $\{P_i\}_{i\in[n]} \subseteq \mathcal{M}$ s.t. $\tau(P_i) = \frac{1}{\lambda}$ and $P_i \wedge (\bigvee_{j:\{i,j\}\notin E} P_j) = 0\ \forall\, i \in [n]$

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5. $\max\{d : K_d \xrightarrow{qc} \overline{G}\} = \alpha_{qc}(G) \leq \Theta_{qc}(G) \leq \mathcal{H}_{tr}(G)$

Properties of the Tracial Haemers Bound

$\mathcal{H}_{tr}(G) = \inf\{\lambda : \exists\, \lambda - \text{tracial subsp. rep. of } G\}$, where a λ-tracial subspace rep. is

- A vN algebra $\mathcal{M}$ with a trace $\tau : \mathcal{M} \to \mathbb{C}$
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$\mathcal{H}_{tr}(G) = \inf\{\lambda : \exists\, \lambda - \text{tracial proj. rep. of } G\}$, where a λ-**tracial projection rep.** is

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- Proj. $\{P_i\}_{i\in[n]} \subseteq \mathcal{M}$ s.t. $\tau(P_i) = \frac{1}{\lambda}$ and $\|P_i(\bigvee_{j:\{i,j\}\notin E} P_j)\| = 0\ \forall\, i \in [n]$

For projections $P, Q \in B(\mathcal{H})$, $\|PQ\| = 0 \Rightarrow P \wedge Q = 0$; $\Leftarrow$ holds when $\mathcal{H}$ is finite-dim.

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Summary and Open Problems

$$
\begin{array}{ccccccccccccc}
\alpha(G) & \leq & \alpha_q(G) & \leq & \alpha_{qc}(G) & & & & & & & & \\
\rotatebox{-90}{$\leq$} & & \rotatebox{-90}{$\leq$} & & \rotatebox{-90}{$\leq$} & & \mathcal{H}_{tr}(G) & \leq & \mathcal{H}_f(G;\mathbb{C}) & & & & \\
\Theta(G) & \leq & \Theta_q(G) & \leq & \Theta_{qc}(G) & \leq & & \not\lessgtr & & \leq & \overline{\xi}_f(G) & \leq & \overline{\chi}_f(G), \\
& & & & & & \vartheta(G) & \leq & \overline{\xi}_{tr}(G) & & & &
\end{array}
$$

- Two new elements in X_{qc}: $\mathcal{H}_{tr}$ and $\overline{\xi}_{tr}$. A better upper bound on Θ_{qc}: $\mathcal{H}_{tr}$;

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- Two new elements in $\mathbf{X}_{qc}$: $\mathcal{H}_{tr}$ and $\overline{\xi}_{tr}$. A better upper bound on Θ_{qc}: $\mathcal{H}_{tr}$;
- A unified descriptions for other elements in $\mathbf{X}$

- $\mathcal{H}_{tr}(G)$+orthogonality $\Rightarrow \overline{\xi}_{tr}(G)$
- $\mathcal{H}_{tr}(G)$+finite-dim. $\Rightarrow \mathcal{H}_f(G;\mathbb{C})$
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- $\mathcal{H}_{tr} = \mathcal{H}_f$ and $\overline{\xi}_{tr} = \overline{\xi}_f$ if the Connes embedding conjecture is true.
- Comm. and Noncomm. ψ hierarchies [Gvozdenović-Laurent 08, GdLL 18]
  - $\vartheta(G) = \psi_1(\overline{G}) \leq \cdots \leq \psi_{\alpha(G)}(\overline{G}) = \cdots = \psi_\infty(\overline{G}) = \overline{\chi}_f(G)$
  - $\vartheta(G) = \xi_1^{\text{col}}(\overline{G}) \leq \cdots \leq \xi_\infty^{\text{col}}(\overline{G}) = \overline{\xi}_{tr}(G) \leq \xi_*^{\text{col}}(G) = \overline{\xi}_f(G)$

  Question 1: Can $\mathcal{H}_{tr}(G)$ and $\mathcal{H}_f(G)$ be approximated by SDP?

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- **Question 2:** Are $\mathcal{H}_{tr}(G)$ and $\mathcal{H}_f(G)$ comparable with $\alpha_*(G)$? (Related to disprove $\alpha_q = \alpha_*$ and $\Theta_* = \vartheta$.)

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Thanks for your attention!